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LETTER TO THE EDITOR

Free field–matter commutation relations in operator radiation reaction theory

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Abstract. Resonant light scattering is considered in the Heisenberg picture, by making use of operator radiation reaction theory. A crucial commutator between matter and free-field variables is shown to vanish, and the validity of the quantum fluctuation-regression theorem, like that of the optical Bloch equations which govern the reduced atomic density matrix, is firmly established, apart from renormalization difficulties, within the rotating wave approximation.

The validity of the optical Bloch equations which govern the density matrix for a driven two-level atom has recently been established by fully quantum mechanical arguments both in the Schrödinger picture (Mollow 1975) and in the Heisenberg picture (Ackerhalt and Eberly 1974, Kimble and Mandel 1975, Saunders *et al* 1975), and has been shown in particular to depend only upon the relatively unrestrictive assumption that the saturated linewidth is small compared to the optical atomic resonance frequency. The same assumption is shown by a Schrödinger-picture argument (Mollow 1975) to guarantee the validity of the quantum fluctuation-regression theorem (Lax 1968), and hence of the result which has been found through its use for the frequency spectrum of resonantly scattered light (Mollow 1969).

The purpose of this letter is to show that the validity of the fluctuation-regression theorem can be established by a relatively simple argument in the Heisenberg picture, where it can be shown (Ackerhalt and Eberly 1974, Kimble and Mandel 1975, Saunders *et al* 1975) that in the dipole approximation, the positive-frequency part $\boldsymbol{E}^{(+)}(\boldsymbol{r} = 0, t)$ of the electric field operator at the position of the atom can be found from the relation

$$\boldsymbol{\mu}_{10} \cdot \boldsymbol{E}^{(+)}(\boldsymbol{r} = 0, t) = \mathrm{i}\hbar(\tfrac{1}{2}\kappa + \mathrm{i}\delta\omega)a_{10}(t) + \boldsymbol{\mu}_{10} \cdot \boldsymbol{E}_{\mathrm{F}}^{(+)}(\boldsymbol{r} = 0, t), \tag{1}$$

in which $\boldsymbol{\mu}_{10}$ is the electric dipole matrix element connecting the upper atomic state $|1\rangle_{\mathrm{a}}$ with the atomic ground state $|0\rangle_{\mathrm{a}}$; κ and $\delta\omega$ are the Einstein A coefficient and the radiative frequency shift, respectively, for the transition in question; $a_{\lambda\nu}(t)$ is the Heisenberg atomic transition operator which has the initial value $a_{\lambda\nu} = |\nu\rangle_{\mathrm{a}\,\mathrm{a}}\langle\lambda|$ (where $\lambda, \nu = 0, 1$); and $\boldsymbol{E}_{\mathrm{F}}(\boldsymbol{r}, t)$ is the freely-propagating part of the electric field operator.

For an initially coherent field-state (Glauber 1963), the validity of the optical Bloch equations which govern the reduced atomic density matrix elements

$$\rho_{\lambda\nu}(t) = \langle |a_{\lambda\nu}(t)| \rangle \tag{2}$$

follows directly (Ackerhalt and Eberly 1974, Kimble and Mandel 1975, Saunders *et al*

1975) from (1) and the relations (Glauber 1963)

$$E_F^{(+)}(\boldsymbol{r}, t)|\rangle = E_c^{(+)}(\boldsymbol{r}, t)|\rangle; \qquad \langle|E_F^{(-)}(\boldsymbol{r}, t) = \langle|E_c^{(-)}(\boldsymbol{r}, t), \tag{3}$$

in which $\boldsymbol{E}_c(\boldsymbol{r}, t)$ is the c-number amplitude of the multimode coherent state, propagated freely in time, and $|\rangle$ is the fixed Heisenberg state vector for the joint atom–field system.

One may seek to establish the validity of the fluctuation-regression theorem (and hence the accuracy of the value found (Mollow 1969) for the atomic correlation function $\langle|a_{10}^\dagger(t')a_{10}(t)|\rangle$ which determines the scattered field spectrum) by similarly showing that the quantities

$$R_{\lambda\nu}(t;t') \equiv \langle|a_{10}^\dagger(t')a_{\lambda\nu}(t)|\rangle \tag{4}$$

(where $t \geqslant t'$) obey the same set of optical Bloch equations as do the quantities $\rho_{\lambda\nu}(t)$. The proof of this is entirely straightforward, but, as noted by Hassan and Bullough (1975), requires the satisfaction of the commutation relation

$$[E_F^{(+)}(0, t), a_{10}(t')] = [a_{10}^\dagger(t'), E_F^{(-)}(0, t)]^\dagger = 0 \qquad \text{for } t \geqslant t'. \tag{5}$$

It can be shown† that the commutator of the free electric field vector $\boldsymbol{E}_F(x)$ and the full Heisenberg current operator $\boldsymbol{J}(x')$ can be expressed quite rigorously and generally at any pair of space–time points $x = \boldsymbol{r}, t$ and $x' = \boldsymbol{r}', t'$ as

$$[\boldsymbol{E}_F(x), \boldsymbol{J}(x')] = -[\mathscr{E}(x), \boldsymbol{J}(x')] \tag{6a}$$

where

$$\mathscr{E}(x) \equiv -\frac{\partial}{\partial t}\int d^4\bar{x}(D_C(x-\bar{x}) - D_A(x-\bar{x}))\theta(t'-\bar{t})\boldsymbol{J}(\bar{x}) \tag{6b}$$

$D_C(x)$ and $D_A(x)$ are the causal and advanced electromagnetic Green functions, respectively, and the unit step function $\theta(t'-\bar{t})$ vanishes unless $\bar{t} < t'$. For $t > t'$, the advanced Green function can make no contribution in (6b), while for $\boldsymbol{r} = 0$ and $t > t'$ the causal Green function can make a contribution only if $c(t-t')$ is less than the maximum radius R of the current distribution. It follows then quite exactly that

$$[\boldsymbol{E}_F(0, t), \boldsymbol{J}(\boldsymbol{r}', t')] \equiv 0 \qquad \text{for } t-t' > R/c. \tag{7}$$

A similar argument, with $\boldsymbol{J}(\boldsymbol{r}', t')$ replaced by any matter-operator $Q_a(t')$, leads to the same conclusion,

$$[\boldsymbol{E}_F(0, t), Q_a(t')] = 0 \qquad \text{for } t-t' > R/c. \tag{8}$$

In the limit $R \to 0$, then, where the dipole approximation is valid, (8) holds for all $t > t'$, and hence the positive- and negative-frequency parts (with respect to t) of the left-hand side of (8) must each certainly vanish after a few optical periods. Since (5) is satisfied identically at $t = t'$ by virtue of (1) (Hassan and Bullough 1975), and since the rotating wave approximation in any case is based upon a time-average over the optical period, the validity of (5) and hence of the fluctuation-regression theorem is proved, within the rotating wave approximation‡.

† See for example Mollow (1973), equations (2.17) and (2.7); the quantum statistical brackets $\langle\ \rangle$ should be deleted from equations (2.17) and (2.16).

‡ One has the option of introducing the rotating wave approximation *after* evaluating the commutator between the atomic and free-field operators in the equations governing $R_{\lambda\nu}(t;t')$. If this is done, the frequency signatures on the free-field operators do not appear in (5), and the relevant commutator then vanishes identically for $t > t'$ by virtue of (8).

Lest it be feared that the limit $R \to 0$ introduces a singularity which affects the validity of the above argument (*such a singularity could affect the Bloch equations for $R_{\lambda\nu}(t;t')$ only at $t = t'$*), it is perhaps worth mentioning that the entire argument can be carried out within the dipole approximation, where one finds the relation

$$[E_{\mathrm{F}}^{(+)}(0,t), Q_{\mathrm{a}}(t')] = -\mathrm{i}\mu_{01}\hbar^{-1}F(t-t')\exp[\mathrm{i}\omega_{10}(t'-t)][a_{10}(t'), Q_{\mathrm{a}}(t')] \qquad \text{for } t \geqslant t', \quad (9)$$

where ω_{10} is the atomic resonance frequency, while $F(t-t')$ is a function (Mollow 1975) which for $t > t'$ falls to zero rapidly (within a few optical periods) with increasing t, and which at $t = t'$ has the finite value $F(0) = \hbar^2(\frac{1}{2}\kappa + \mathrm{i}\delta\omega)/|\mu_{10}|^2$. Equation (5) then follows immediately upon setting $Q_{\mathrm{a}}(t') = a_{10}(t')$ in (9).

The radiative frequency shift $\delta\omega$ of course is finite only if a cut-off, for example, is introduced into the photon density of states (Saunders *et al* 1975), but the same problem arises in deriving the (optical Bloch) equations which govern the atomic density matrix. The equations governing the two-time atomic correlation functions thus enjoy the same degree of validity as do the (same) equations governing the atomic density matrix, both being firmly and equally established, apart from renormalization questions, within the rotating wave approximation.

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